Automated exploration of envelopes

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Abstract:

The study of envelopes is a classical topic. One of its characteristics is the small number of theorems, therefore the study of envelopes of families of plane curves and of surfaces in 3D space is mostly made of the study of special cases.

Tools for automated study of these objects have been developed during the last years, in Dynamical Geometry Systems (DGS), for example in GeoGebra. A joint usage of the dynamical features (slider bar, mouse dragging) and of implemented commands (**Envelope**(<Path>,<Point>) may yields double answers: graphical – a plot of the desired envelope, and algebraic – an implicit equation for the envelope. A graphical answer may provide a conviction that an envelope exists, without being a full proof. Of course, the family of curves (resp. surfaces has to be defined in a format suitable for the command to be effective).

Nevertheless, this kind of double answer is not always available, for programming reasons. Another kind of software, namely a Computer Algebra System (CAS), may be useful to compute a parametric presentation of the desired envelope (a result who is a proof of the existence of the envelope). Further computing with algebraic packages (heavy algebraic machinery is sometimes needed, e.g. in order to work with polynomials, enabling to apply algorithms from the theory of Gröbner bases) may yield an implicit equation from the envelope, but not always.

Then back to the DGS an accurate plot of the envelope is obtained, providing a confirmation of the conjecture established in first part.

We illustrate this kind of exploration with examples in 2D having the following characteristics:

1. The dynamical plot with DGS gives a good intuitive plot (a family of circles centered on an ellipse)
2. The choice of two different dynamical tools provide different plots, one of them much more useful to establish a conjecture (nephroid).
3. An envelope is often thought as a "wrap" for the given family. The third example will be non-intuitive (a family of circles centered on an astroid).

When computing an analytic presentation for the envelope, the CAS provides often a description of the envelope as the union of (disjoint) components. The

examples will emphasize the respective roles of these components.