Teaching Programming to Mathematical Scientists

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We live in a rough world

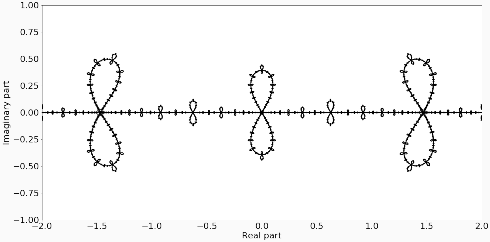

Figure 1: An infinite number of infinity symbols

"The existence of these patterns [fractals] challenges us to study forms that Euclid leaves aside as being formless, to investigate the morphology of the amorphous." (Benoit B. Mandelbrot, *The Fractal Geometry of Nature*, 1983, p. 1)

Context

- Teaching *Computational Mathematics* is increasingly important (Data Science, Visualization, Machine Learning, . . .)
- This is *difficult* because computational mathematics involves several things at once: mathematics, programming, complexity, and numerical stability, because of the *compromises* needed for efficiency.
- **Incorporating new things means removing old things** because we have only finite time to teach, and the students are learning other things as well.
- “A spoon-ful of sugar helps the medicine go down”

Fractals are sweet

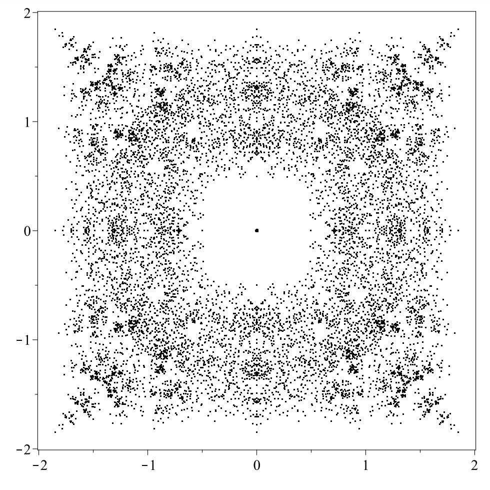

Figure 2: A Bohemian Example: All eigenvalues computed numerically by Maple of all 4096 seven by seven skew-symmetric bidiagonal matrices with entries from $\{1, i, 1+i, 1-i\}$. See bohemianmatrices.com

Pass the Parcel (avanzar el paquete)

Newton's iteration is $z_{n+1} = z_n - F(z_n)/F'(z_n)$. To explain this to students we play the game of "pass the parcel": given an initial function (e.g. $F(z) = z^3 - 1$).

- One student chooses an initial number z_0 and passes it to the next. Here $n = 0$.
- The receiving student computes the next number by $z_{n+1} = z_n - F(z_n)/F'(z_n)$ and passes that result to the succeeding student.
- We go around the room until the iteration converges, or we get bored, or everyone has had a chance.

Mandelbrot polynomials

```
function [p,dp]=mandelpoly(z,k)
% MANDELPOLY evaluates the k^th Mandelbrot polynomial
% and its derivative at one or more points.
% The k^th polynomial has degree 2^(k-1)-1

% Author PWL 2014.4.28 Modified RMC 2020.2.27
    dp = zeros(size(z));
    p  = zeros(size(z));
    for i=1:k-1
        dp = p.^2+2*z.*p.*dp;
        p  = z.*p.^2+1;
    end
```

The sixth iterate, $p_6(x)$, for real x

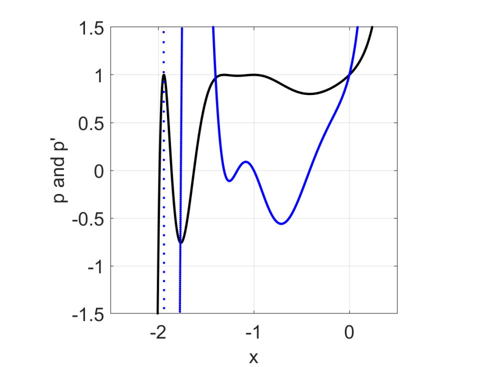

Figure 3: Graph of $p_6(x)$ and its derivative (blue) $p_6'(x)$ where $p_0(x) = 0$ and $p_{n+1}(x) = xp_n^2(x) + 1$.

Newton Fractals

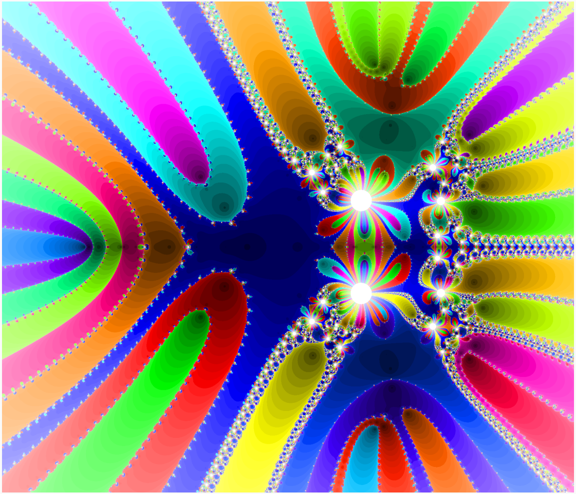

Figure 4: Newton fractal of $p_6(z)$ (generated using Python)

Student-generated fractals

Figure 5: Forty student-generated fractals

Challenges from IEEE floats

- "Admit, for instance, the existence of a minimum magnitude, and you will find that the minimum which you have introduced, small as it is, causes the greatest truths of mathematics to totter." — Aristotle
- Floats are *not associative*: $a + (b + c) \neq (a + b) + c$ necessarily. For instance $-M + (M + 1) = 0$ while $(-M + M) + 1 = 1$ if $M = 3.14 \cdot 10^{17}$.
- This (and other features) break students' models of how the world works.
- We have to enable students to deal with floats.

Mathematical Notions Strengthened by Programming

Several mathematical notions are strengthened by these exercises.

- We use *mathematical induction* to prove correctness of the automatic differentiation of the Mandelbrot polynomials
- The analysis of IEEE floats uses the IEEE guarantees ($\mathrm{fl}(x \mathbin{\mathrm{op}} y) = (x \mathbin{\mathrm{op}} y)(1+\delta)$ for some $|\delta| \le \mathbf{u}$ where $\mathbf{u}$ is the *unit roundoff*, 2^{-53} for double precision)
- Practice with functions is always useful
- Simply working with visualizations improves people's feel for geometry.

Reference Papers

Chauvenet prize-winning papers

1. Wilkinson, James H. “The perfidious polynomial.” Studies in numerical analysis 24 (1984): 1-28.
2. Olds, C. D. “The simple continued fraction expansion of *e*.” The American Mathematical Monthly 77.9 (1970): 968-974.
3. Davis, Philip J. “Leonhard Euler’s integral: A historical profile of the Gamma function: In memoriam: Milton Abramowitz.” The American Mathematical Monthly 66.10 (1959): 849-869.

Reference Texts and Videos

1. Moler, Cleve B. Numerical Computing with MATLAB: Revised Reprint. Vol. 87. Siam, 2008.
2. Higham, Desmond J., and Nicholas J. Higham. MATLAB guide. Society for Industrial and Applied Mathematics, 2016.
3. MIT OpenCourseWare: Gil Strang and Cleve Moler

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- Corless, Robert M. ”Computer-mediated thinking.” Proceedings of Technology in Mathematics Education (2004).
- Frame, Michael, and Benoit B. Mandelbrot. Fractals, graphics, and mathematics education. No. 58. Cambridge University Press, 2002.
- Li, Ao, and Robert M. Corless. “Revisiting Gilbert Strang’s *A chaotic search for i*.” ACM Communications in Computer Algebra 53.1 (2019): 1-22.

¡Time for Questions!