Automated exploration of envelopes

animation vs exploration - the necessary dialog between technologies

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צולם ביום חול

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General definition of the envelope of a family of plane curves

Consider a parameterized family *F* of plane curves , dependent on a real parameter *k*. A plane curve *E* is called **an envelope** of the family *F* if the following properties hold:

(i) every curve is tangent to *E*;

(ii) to every point *M* on *E* is associated a value *k*(*M*) of the parameter *k*, such that is tangent to *E* at the point *M*;

(iii) The function *k*(*M*) is non-constant on every arc of *E*.

- Kock: *Impredicative* definition

Kock. A. (2007) Envelopes - notion and definiteness, Beiträge zur Algebra und Geometrie (Contributions to Algebra and Geometry) 48, 345-350.

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First example: algebraic treatment

We consider the 1-parameter family *F* of lines given by the equations

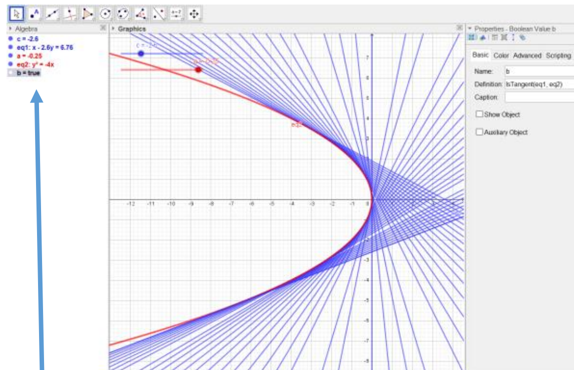
$$x + cy = c^2$$

where *c* is a real parameter.

1. We conjecture **that an envelope is the parabola whose equation is**

$$x = -\frac{1}{4}y^2$$

2. **We check this graphically.**
3. **We check this algebraically.**



IsTangent

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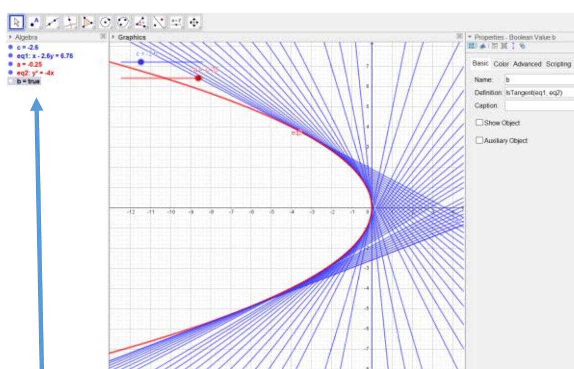

IsTangent

Exploration
Intuition
Conjecture
Checking

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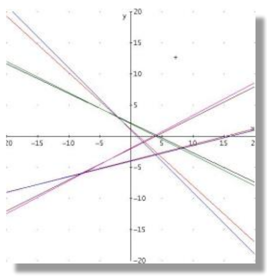
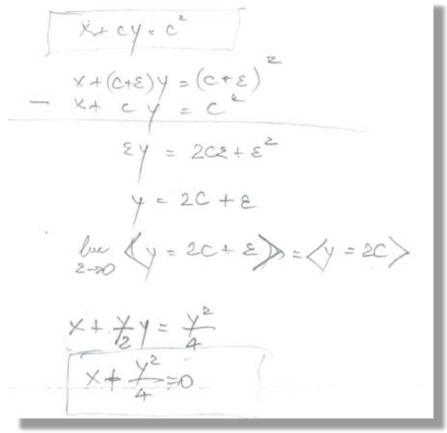
Same example: infinitesimally close lines

We consider the 1-parameter family *F* of lines given by the equations $x + cy = c^2$

where *c* is a real parameter.





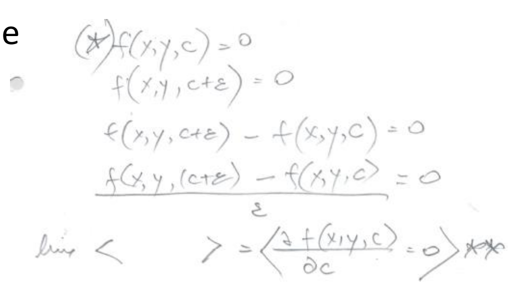
Solving the system of equations [f(x,y,c)=0 and der(f(x,y,c),c)=0]

An envelope of the family is (a *subset* of) the curve defined by the following equations:

$$\begin{cases} f(x,y,c) = 0 \\ \dfrac{\partial}{\partial c} f(x,y,c) = 0 \end{cases}$$

Kock: analytic definition

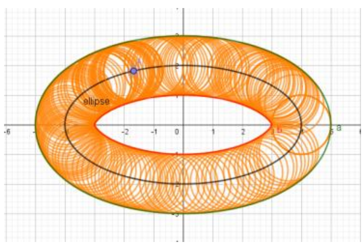
- Proof:



Circles centered on a closed curve

On an ellipse

- ellipse: $0.25x^2 + y^2 = 4$
- $A = (-1.68, 1.82)$
- c: $(x + 1.68)^2 + (y - 1.82)^2 = 1$
- a: $\begin{cases} x = \cos(t)\,\dfrac{2 \cdot \frac{\sqrt{3\sin^2(t)+1}+1}{\sqrt{3\sin^2(t)+1}} + 6}{2} \\ y = 2\left(\sqrt{3\,\sin^2(t)+1}+1\right)\dfrac{\sin(t)}{\sqrt{3\,\sin^2(t)+1}} \end{cases} \quad 0 \le t \le 6.28$
- b: $\begin{cases} x = \cos(t)\,\dfrac{2 \cdot \frac{\sqrt{3\sin^2(t)+1}-1}{\sqrt{3\sin^2(t)+1}} + 6}{2} \\ y = 2\left(\sqrt{3\,\sin^2(t)+1}-1\right)\dfrac{\sin(t)}{\sqrt{3\,\sin^2(t)+1}} \end{cases} \quad 0 \le t \le 6.28$



- ellipse: $0.25x^2 + y^2 = 4$
- $A = (0.29, 1.99)$
- c: $(x - 0.29)^2 + (y - 1.99)^2 = 4$
- a: $\begin{cases} x = \cos(t)\,\dfrac{2 \cdot \frac{\sqrt{4-3\cos^2(t)}-2}{\sqrt{4-3\cos^2(t)}} + 6}{2} \\ y = 2\left(\sqrt{4-3\,\cos^2(t)}-2\right)\dfrac{\sin(t)}{\sqrt{4-3\,\cos^2(t)}} \end{cases} \quad 0 \le t \le 6.28$
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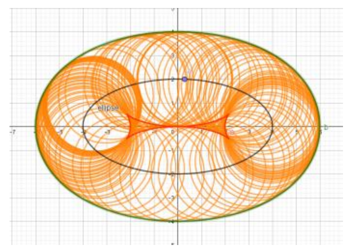

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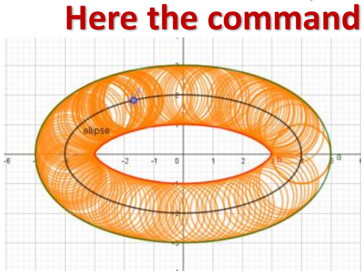
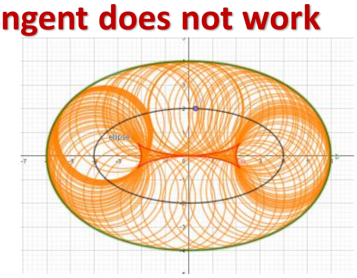
Here the command IsTangent does not work





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What can happen?

- If the command IsTangent does not work:

  Check tangency by algebraic/analytic means

- If the Envelope command does not work:

Solve the system of equations

  Check intersection of the given curve with the arcs that have been obtained

  Check the multiplicity of contact

Circles centered on a closed curve

$restart : with(plots) :$

$ellcurve := \frac{x^2}{16} + \frac{y^2}{4} - 1;$

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$c := (x - 4\cdot\cos(t))^2 + (y - 2\sin(t))^2 - 4;$

$$c := (x - 4\cos(t))^2 + (y - 2\sin(t))^2 - 4 \qquad (2)$$

$derc := \frac{\partial}{\partial t}(c);$

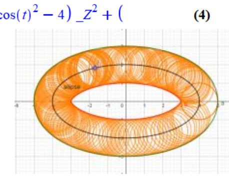
$$derc := 8\,(x - 4\cos(t))\sin(t) - 4\,(y - 2\sin(t))\cos(t) \qquad (3)$$

$solve(\{c = 0, derc = 0\}, \{x, y\});$

$$\left\{x = \frac{\cos(t)\left(RootOf\left(\left(3\cos(t)^2 - 4\right)\_Z^2 + \left(-12\cos(t)^2 + 16\right)\_Z + 12\cos(t)^2\right) + 6\right)}{2}, y = RootOf\left(\left(3\cos(t)^2 - 4\right)\_Z^2 + \left(-12\cos(t)^2 + 16\right)\_Z + 12\cos(t)^2\right)\sin(t)\right\} \qquad (4)$$

$allvalues(\%);$

$$\left\{x = \frac{\cos(t)\left(\frac{2\left(\sqrt{4 - 3\cos(t)^2} - 2\right)}{\sqrt{4 - 3\cos(t)^2}} + 6\right)}{2}, y = \frac{2\left(\sqrt{4 - 3\cos(t)^2} - 2\right)\sin(t)}{\sqrt{4 - 3\cos(t)^2}}\right\}, \left\{x = \frac{\cos(t)\left(\frac{2\left(\sqrt{4 - 3\cos(t)^2} + 2\right)}{\sqrt{4 - 3\cos(t)^2}} + 6\right)}{2}, y = \frac{2\left(\sqrt{4 - 3\cos(t)^2} + 2\right)\sin(t)}{\sqrt{4 - 3\cos(t)^2}}\right\}$$



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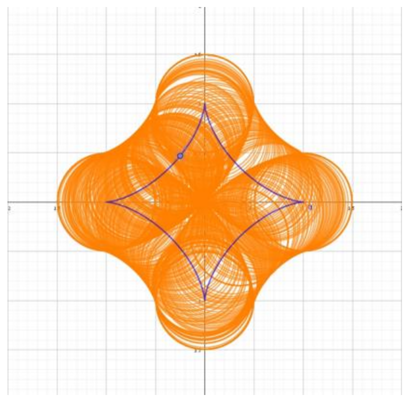
b: $x = \cos(t)\,\frac{2\cdot\frac{\sqrt{3\sin^2(t)+1}-1}{\sqrt{3\sin^2(t)+1}} + 6}{2}$, $y = 2\left(\sqrt{3\sin^2(t)+1} - 1\right)\frac{\sin(t)}{\sqrt{3\sin^2(t)+1}}$, $0 \le t \le 6.28$

Maple 2019

GeoGebra

Intuition vs computations

- Astroid:
  - Implicit equation
  - Parametric presentation
- Family of circles centered on the astroid, with radius 1/2
- Intuition: there is an envelope, namely the curve circumscribing the colored zone



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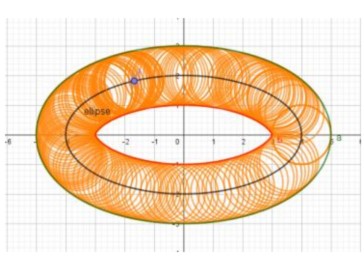

On an astroid

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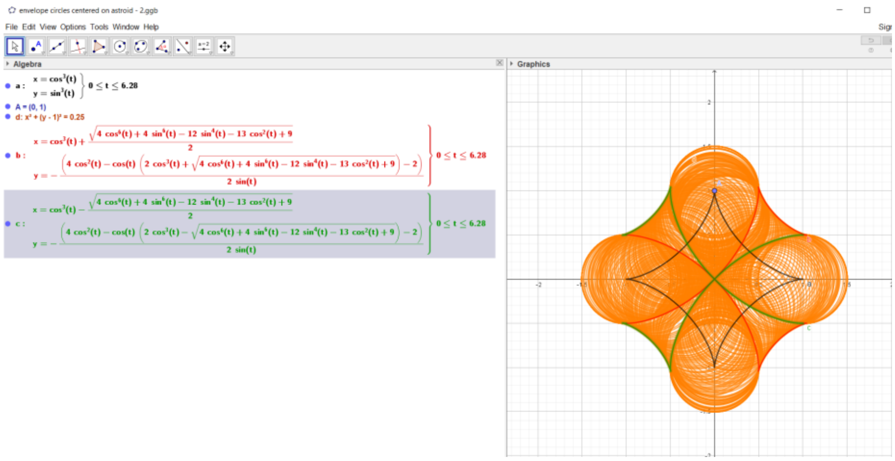
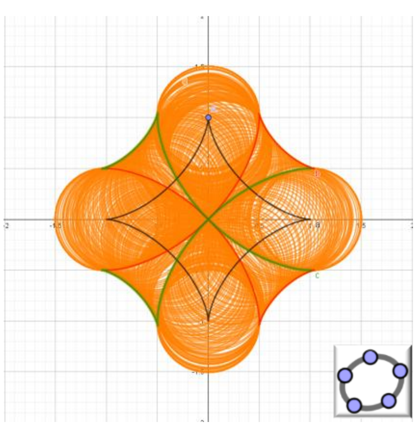
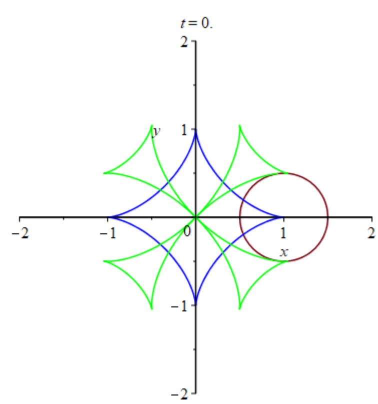
Circles centered on an astroid

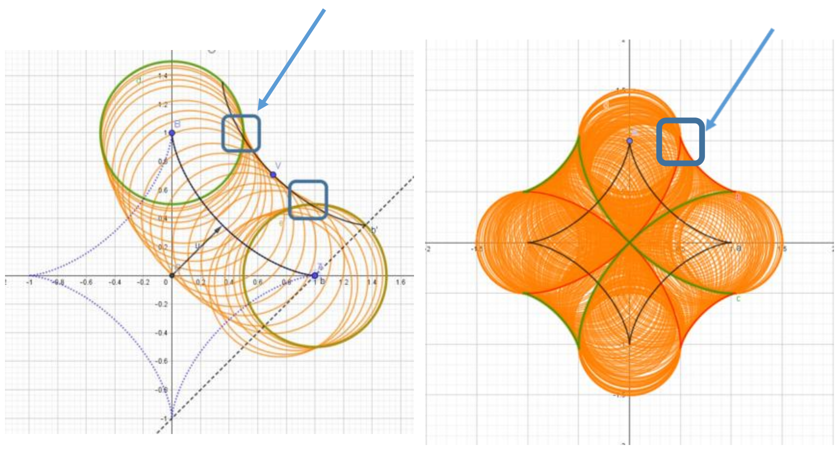


Animation vs interactive exploration





Problems with intuition



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Some general conclusions

- Intuition does not always fit the situation. **TRIVIAL!!!**
- Necessary dialog between DGS and CAS: it may go with copy-paste. Maybe in the future ….
- The dialog goes really in both directions.
- Switching between registers of representation for mathematical objects (Duval, Presmeg, etc.)
  - Paper and pencil work
  - Within a single package (generally a CAS)
  - **We upgrade the switches: switching goes in reversed directions and between two different kinds of software**

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Some references

Th. Dana-Picard and N. Zehavi (2016): *Revival of a classical topic in Differential Geometry: the exploration of envelopes in a computerized environment*, International Journal of Mathematical Education in Science and Technology 47(6), 938-959.

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